Conformally Invariant Gauge Theory of 3-Branes in 6-D and the Cosmological Constant

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It is shown that the gauge theory of relativistic 3-Branes can be formulated in a conformally invariant way if the embedding space is six-dimensional. The implementation of conformal invariance requires the use of a modified measure, independent of the metric in the action. Brane-world scenarios without the need of a cosmological constant in $6D$ are constructed. Thus, no "old" cosmological constant problem appears at this level.

I. INTRODUCTION

Extended objects of various dimensions are present in the modern formulation of string theory. Among the various kind of branes a unique role is played by D–branes [1] as they can trap the end-points of open strings. D-brane inspired cosmological models, commonly termed "brane–universe" models, are currently under investigation as they seem to offer a possible solution to the longstanding hierarchy problem in gauge theories.

We think that the gauge theory formulation of p–branes, proposed some years ago as an alternative to the standard description of relativistic extended objects [2],[3], is well suited to describe this new type of cosmological scenario. Furthermore, the description of p–branes in terms of associated gauge potentials offers a vantage point to study some specific problem as the one concerning the fine tuning of the cosmological constant.

Here we will see that for 3-branes considered in an embedding $6D$ space the gauge theory formulation of 3-branes allows a conformally invariant realization. An essential element necessary to implement conformal invariance is the introduction of a measure of integration in the action which is independent of the metric [4],[5],[6], [7]. We use then such a formulation to construct a new type of brane world scenario.

Brane world scenarios in general are concerned with the possibility that our universe is built out of one or more 3-branes living in some higher dimensional space, plus some bulk component [8], [9], [10], [11], [12]. In particular, the possibility of 3-branes embedded in $6D$ space has been studied in [13], [14], [15], [16], [17]. In this case the effect of the tension of the branes is to induce curvature only in the extra dimensions. In these models there is still a question of fine tuning that has to be addressed, since although the branes themselves do not curve the observed four dimensions, the bulk components of matter do, and they have to be fine tuned in order to get (almost)zero four dimensional vacuum energy. This very special feature of 3-branes is a $6D$ embedding spacetime is related to the fact that such matter content, even coupled to gravity, has a conformal invariance associated to it.

The strategy that we will follow in order to solve this problem is to incorporate the "brane-like features" that are quite good in what concerns the cosmological constant problem into the "bulk" part of the brane scenario as well. In this way both bulk and singular brane contributions will share the fundamental feature of curving only the extra dimensions. The gauge field formulation of 3-branes in $6D$ (GFF3B6D) is ideally suited for such a program.

As we will see GFF3B6D allows us to understand , extend and give a "pure brane interpretation" of the results of [21], where a "square root gauge theory", coupled to 3-branes in $6D$ was considered. This model has conformal invariance and there is no need to introduce a $6D$ cosmological constant. The "fundamental physics" behind the model is not so clear however and its different matter elements: gauge fields, 3-branes appear rather disconnected from each other. We will see that GFF3B6D allows us resolve these drawbacks and present a more general set of brane-world solutions, where the solutions presented in [21] appear as very particular cases. The paper is organized as follows. In Sect.II, we go through the gauge formulation of branes; in Sect.III, we study the GFFB6D and show that this system displays conformal invariance when a modified measure is introduced, also the dual picture to this formulation is introduced; in Sect.IV, the equations of motion in this dual picture are studied. We end up with a brief discussion and conclusions.

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II. SUMMARY OF GAUGE FORMULATION OF BRANES

In this section we would like to recall some features of the "gauge formulation" of brane dynamics introduced some years ago by one of the authors (E.S.). A possible way to bridge the gap between extended objects and gauge field is to provide a non-geometric description of the former ones in terms of appropriate field variables. This model has been investigated in depth in [2], [3]. Here we do not report about what has been already published but add some new matter related to D-branes.
The effective description of extended objects in terms higher rank gauge forms requires the introduction of both a *slope field* and a *geodesic field*. The slope field defines the tangent (hyper)plane to brane world surface in target spacetime. The geodesic field represents the brane field strength. As such it solves a set of Bianchi identities which reproduce the classical equation of motion once are projected on the brane world surface with the aid of the geodesic field.

A. Brane slope field and gauge potential.

A p-brane can be described as $p+1$ dimensional hypersurface embedded into an higher D-dimensional target spacetime. The parametric equations describing the brane world-volume reads $x^M = Y^M\left(\sigma^0 ,\vec{\sigma}\right)$, where $\vec{\sigma} = \left(\sigma^0 ,\sigma^1 ,\dots ,\sigma^p\right)$, with $p+1 \le 1+d \equiv D$ and $x^M = \left(x^0 ,x^1 ,\dots ,\right)$ are target spacetime coordinates. To each point of the brane world-volume one can define a tangent hyperplane. The orientation of the tangent hyperplane is encoded into the current density

$$J^{M_1\dots M_{p+1}}(x) = \int d^{p+1}\sigma\, \delta^{(d+1)}\left[\, x - Y(\sigma)\,\right] \epsilon^{m_1\, m_2\, \dots\, m_{p+1}}\partial_{m_1}Y^{M_1}\dots\partial_{m_{p+1}}Y^{M_{p+1}} \tag{1}$$

The Dirac-delta makes J to be "singular" , i.e. J different from zero only along the brane world-volume. However, we can define a "*slope field*" $F^{M_1\dots M_{p+1}}(x)$ as a "regular" field whose restriction on the brane world-volume matches the the tangent hyperplane:

$$\int d^{d+1}x\, \delta^{(d+1)}\left[\, x - Y(\sigma)\,\right] F^{M_1\dots M_{p+1}}\left(\, x\,\right) = \epsilon^{m_1\, m_2\, \dots\, m_{p+1}}\partial_{m_1}Y^{M_1}\dots\partial_{m_{p+1}}Y^{M_{p+1}} \tag{2}$$

Accordingly, the current density can be written as

$$J^{M_1\dots M_{p+1}}(x) = \int d^{p+1}\sigma\, \delta^{(d+1)}\left[\, x - Y(\sigma)\,\right]\, F^{M_1\dots M_{p+1}}\left(\, x\,\right) \tag{3}$$

A gauge-type formulation of brane dynamics can be recovered from the action

$$\begin{aligned} S = \quad & e^2 \int d^{d+1}x\, \sqrt{-g_{(d+1)}}\left[-\frac{1}{(p+1)!}\, g_{M_1\, N_1}\dots g_{M_{p+1}\, N_{p+1}}\, W^{M_1\dots M_{p+1}} W^{N_1\dots N_{p+1}}\right]^{1/2} \\ & + \frac{1}{(p+1)!}\int d^{d+1}x\, \sqrt{-g_{(d+1)}}\, W^{M_1\dots M_{p+1}}\, \partial_{[\, M_1}\, B_{M_2\dots M_{p+1}\,]} \end{aligned} \tag{4}$$

where W is a contravariant, totally antisymmetric tensor, and the curl of the gauge potential B is the geodesic field we mentioned above. A target spacetime metric $g_{M_1\, M_2}$ has been introduced to make the model generally covariant. e^2 is a constant with mass-squared dimension providing to W the canonical dimension of a gauge field strength in a $d+1$ dimensional spacetime. The B-field plays the role of Lagrange multiplier which imposes W to be divergence-free:

$$\frac{\delta S}{\delta B_{M_2\dots M_{p+1}}} = 0 \longrightarrow \partial_M\, \left(\sqrt{-g_{(d+1)}}\, W^{M_1\dots M_{p+1}}\right) = 0 \tag{5}$$

The divergence in (5) is a generally covariant operator built up with the aid of an appropriate affine connection. The field equation (5) has a general solution of the form

$$W^{M_1 \dots M_{p+1}} = \frac{1}{(D-p-1)!} \frac{1}{\sqrt{-g_{(d+1)}}} \epsilon^{M\, M_2 \dots M_{p+1}\, M_{p+2} \dots M_D} \partial_{[\, M_{p+2}} A_{M_{p+3} \dots M_D\,]} + \frac{e^{p-1}}{\sqrt{-g_{(d+1)}}} J^{M\, M_2 \dots M_{p+1}} \tag{6}$$

The "regular" part of W has been written in terms of a $D-p-1$ gauge potential, while the "singular" part is expressed in terms of the current density. The constant e^{p-1} provides the matching between the (different) canonical dimensions of W and J. In order to satisfy equation (6) J must be divergence-free. The condition $\partial_M J^{M\, M_2 \dots M_{p+1}} = 0$ selects boundary-free branes, i.e. closed or infinitely extended objects. Having in mind a cosmological scenario where branes are orthogonal to the extra dimensions, we specialize our model in the following way:
i) $Y^\mu(\sigma)$ are the embedding functions of a D_p–brane, i.e. a static topological defect in the spacetime fabric located at a definite position;
ii) only gravity is free to propagate-off the brane. Thus, the regular part of the W-field must be zero, the "singular" part of W being strictly localized on the brane. Hence, $W = J$.

B. Relation with the world-volume dynamics

After the brief discussion of the gauge formulation of p-brane, summarized above, it may be worth to provide the relations to the more familiar world volume description of p-brane dynamics.
Let us compute the classical action for the solution (6).

$$\begin{aligned} S\,[\,J\,] = &\; e^2 \int d^{d+1}x\, \sqrt{-g_{(d+1)}} \left[-\frac{1}{(p+1)!\left(\sqrt{-g_{(d+1)}}\right)^2}\, e^{2p-2}\, g_{M_1\, N_1} \dots g_{M_{p+1}\, N_{p+1}} J^{M_1 \dots M_{p+1}} J^{N_1 \dots N_{p+1}} \right]^{1/2} \\ = &\; e^{p+1} \int d^{d+1}x \left[-\frac{1}{(p+1)!} \int d^{p+1}\sigma\, \delta^{(d+1)}\,[\, x - Y(\sigma)\,]\, \epsilon^{m_1 \dots m_{p+1}} \partial_{m_1} Y^{M_1} \dots \partial_{m_{p+1}} Y^{M_{p+1}} \right. \times \\ &\left. g_{M_1\, N_1} \dots g_{M_{p+1}\, N_{p+1}} \int d^{p+1}\tilde\sigma\, \delta^{(d+1)}\,[\, x - Y(\tilde\sigma)\,]\, \epsilon^{n_1 \dots n_{p+1}}\; \partial_{n_1} Y^{N_1} \dots \partial_{n_{p+1}} Y^{N_{p+1}} \right]^{1/2} \\ = &\; e^{p+1} \int d^{d+1}x \sqrt{-\frac{1}{(p+1)!} F^{M_1 \dots M_{p+1}} F_{M_1 \dots M_{p+1}}} \times \\ &\left[\int d^{p+1}\sigma\, \delta^{(d+1)}\; [\, x - Y(\sigma)\,] \int d^{p+1}\tilde\sigma\, \delta^{(d+1)}\,[\, x - Y(\tilde\sigma)\,] \right]^{1/2} \end{aligned} \tag{7}$$

Invariance under reparametrization of the brane world volume allows us to write

$$\begin{aligned} \left[\int d^{p+1}\sigma\, \delta^{(d+1)}\,[\, x - Y(\sigma)\,] \int d^{p+1}\tilde\sigma\, \delta^{(d+1)}\,[\, x - Y(\tilde\sigma)\,] \right]^{1/2} &= \left[\left(\int d^{p+1}\sigma\, \delta^{(d+1)}\,[\, x - Y(\sigma)\,] \right)^2 \right]^{1/2} \\ &= \int d^{p+1}\sigma\, \delta^{(d+1)}\,[\, x - Y(\sigma)\,] \end{aligned} \tag{8}$$

Thus,

$$\begin{aligned} S\,[\,J\,] = &\; e^{p+1} \int d^{d+1}x \sqrt{-\frac{1}{(p+1)!} F^{M_1 \dots M_{p+1}} F_{M_1 \dots M_{p+1}}} \int d^{p+1}\sigma\, \delta^{(d+1)}\,[\, x - Y(\sigma)\,] \\ = &\; e^{p+1} \int d^{d+1}x \int d^{p+1}\sigma\, \delta^{(d+1)}\,[\, x - Y(\sigma)\,] \sqrt{-\frac{1}{(p+1)!} F^{M_1 \dots M_{p+1}} F_{M_1 \dots M_{p+1}}} \\ = &\; e^{p+1} \int d^{p+1}\sigma \left[-\frac{1}{(p+1)!} \epsilon^{m_1 \dots m_{p+1}} \partial_{m_1} Y^{M_1} \dots \partial_{m_{p+1}} Y^{M_{p+1}} \epsilon^{n_1 , \dots n_{p+1}} \partial_{n_1} Y^{N_1} \dots \partial_{n_{p+1}} Y^{N_{p+1}} \right]^{1/2} \end{aligned} \tag{9}$$

By introducing the induced metric γ_{mn} as

$$\gamma_{mn}\,(\,\sigma\,) \equiv g_{MN}\,\partial_m\,Y^M\,\partial_n\,Y^N \tag{10}$$

we can write the argument of the square root in (9) in a more transparent form.

$$\begin{aligned}\epsilon^{m_1\,\dots\,m_{p+1}}\partial_{m_1}Y^{^{M_1}}\dots\partial_{m_{p+1}}Y^{^{M_{p+1}}}\epsilon^{n_1\,\dots\,n_{p+1}}\partial_{n_1}Y^{^{N_1}}\dots\partial_{n_{p+1}}Y^{^{N_{p+1}}} &= \epsilon^{m_1\,\dots\,m_{p+1}}\epsilon^{n_1\,\dots\,n_{p+1}}\,\gamma_{m_1\,n_1}\,\cdots\,\gamma_{m_{p+1}\,n_{p+1}}\\ &= (p+1)!\mathrm{Det}\,(\,\gamma_{mn}\,)\end{aligned} \tag{11}$$

Accordingly, the classical action turns out to be just the standard world volume action

$$S = e^{p+1}\int d^{p+1}\sigma\,\sqrt{-\mathrm{Det}\,(\,\gamma_{mn}\,)} \tag{12}$$

where e^{p+1} plays the role of brane tension.
The induced dynamics of a D_p-branes is encoded into a more complex effective action, i.e. a Born-Infeld type functional [18]

$$S = \mathrm{Tension}\int d^{p+1}\sigma\,\sqrt{-\mathrm{Det}\left(\gamma_{mn} + \frac{1}{e^2}F_{mn}\right)} \tag{13}$$

where, $F_{mn} = \partial_{[\,m}\,A_{n\,]}$ is the field strength of a $U\,(\,1\,)$ gauge field. Dirichlet branes fit in our gauge-type description as well. We notice that the Born-Infeld action (13) can be recovered in the same way we obtained (12) provided one replaces $g_{_{M_1\,N_1}}$ in (4) according with the rule

$$g_{_{M_1\,N_1}} \longrightarrow g_{_{M_1\,N_1}} + \frac{1}{e^2}F_{_{M_1\,N_1}} \tag{14}$$

where, $F_{_{M_1\,N_1}}$ is a target spacetime field related to F_{mn} by

$$F_{mn} = F_{_{M_1\,N_1}}\,\partial_m\,Y^{^{M_1}}\,\partial_n\,Y^{^{N_1}} \tag{15}$$

We shall not elaborate anymore on this result which is not essential for what follows. In the next sections we shall drop out the term F/e^2 and focus on the gravitational effects only.
Finally, we recall that the special case $p = 3$ can be made Weyl invariant through the introduction of the *dilaton* field [19]. This field acts as a Stückelberg conpensator [20] under local rescaling of various fields . In the next section we shall describe an alternative, *dynamical* mechanism, to make the D_3-brane action Weyl invariant. Instead of a *fundamental* scalar degree of freedom we shall consider a “composite” scalar density replacing the gravitational volume term $\sqrt{-g_{(5+1)}}$.

C. D-brane gauge formulation

The parametric equations for a D_p–brane are usually written by splitting the coordinates into “parallel” , and “transverse” :

$$x^M = \left(\,x^\mu_{||}\,,x^k_\perp\,\right) \tag{16}$$

$$x^\mu_{||} = Y^\mu_{||}\left(\,\sigma^0\,,\vec{\sigma}\,\right) \tag{17}$$

$$x^k_\perp = Y^k_\perp\,(\,\vec{\sigma}\,) \tag{18}$$

$$0 \le \mu \le p+1\,, \quad p+2 \le k \le D \tag{19}$$

As a first example, let us consider a flat “slab” at the origin of transverse space, that is we choose

$$Y_{\perp}^{k}\left(\,\vec{\sigma}\,\right)=0 \tag{20}$$

and $g_{\mu\nu}^{(3+1)}=\eta_{\mu\nu}$. The D_p–brane current density is of the form

$$\begin{aligned}
J^{M_1\dots M_{p+1}} &= \int d^{p+1}\sigma\,\delta^{(p+1)}\left[x_{||}^{\mu}-Y_{||}^{\mu}\left(\sigma^0\,,\vec{\sigma}\right)\right]\,\delta^{(d-p)}\left(\vec{x}_{\perp}\right)\times\\
&\quad \delta_{\mu_1}^{M_1}\dots\delta_{\mu_{p+1}}^{M_{p+1}}\,\epsilon^{m_1\dots m_{p+1}}\partial_{m_1}Y_{||}^{\mu_1}\dots\partial_{m_{p+1}}Y_{||}^{\mu_{p+1}}\\
&= \delta_{\mu_1}^{M_1}\dots\delta_{\mu_{p+1}}^{M_{p+1}}\delta^{(d-p)}\left(\vec{x}_{\perp}\right)\int dY_{||}^{\mu_1}\wedge\dots\wedge dY_{||}^{\mu_{p+1}}\delta^{(p+1)}\left[x_{||}^{\mu}-Y_{||}^{\mu}\left(\sigma^0\,,\vec{\sigma}\right)\right]\\
&= \delta_{\mu_1}^{M_1}\dots\delta_{\mu_{p+1}}^{M_{p+1}}\delta^{(d-p)}\left(\vec{x}_{\perp}\right)\,J^{\mu_1\dots\mu_{p+1}}\left(x_{||}\right)
\end{aligned} \tag{21}$$

It is immediate to generalize the current to the case where a second D_p–brane is present in different point, say $\vec{x}_{\perp}=\vec{Y}_{\perp}$. Then,

$$J^{M_1\dots M_{p+1}}=\delta_{\mu_1}^{M_1}\dots\delta_{\mu_{p+1}}^{M_{p+1}}\left[\delta^{(d-p)}\left(\vec{x}_{\perp}\right)+\delta^{(d-p)}\left(\vec{x}_{\perp}-\vec{Y}_{\perp}\right)\right]J^{\mu_1\dots\mu_{p+1}}\left(x_{||}\right) \tag{22}$$

By iterating this procedure we can add as many D-branes we want and even to consider a continuous distribution of them. In the latter case

$$J^{M_1\dots M_{p+1}}=\delta_{\mu_1}\dots\delta_{\mu_{p+1}}^{M_{p+1}}\,\rho\left(\vec{x}_{\perp}\right)J^{\mu_1\dots\mu_{p+1}}\left(x_{||}\right) \tag{23}$$

where $\rho\left(\vec{x}_{\perp}\right)$ represents the density of D_p–branes in the transverse space.
In the original papers [2], [3] gravity was introduced by making the action (4) generally covariant and by adding an Einstein term. This procedure will be generalized in the next section in such a way to endow the model with conformal invariance, possible in the case of 3-branes embedded in $6D$ spacetime.

III. CONFORMALLY INVARIANT REALIZATION IN $6D$

Working now in a generally covariant theory where spacetime is $5+1$ dimensional, the relativistic object we would like to consider is a 3-brane. Furthermore, we would like to make the whole model *conformally invariant*. This can be obtained provided we insert a "composite" scalar density Φ in place of $\sqrt{-g_{(5+1)}}$ in the first term in (4) plus a similar contribution in the curvature term in the action

$$\begin{aligned}
S= &-\frac{1}{16\pi G_{(5+1)}}\int d^{5+1}x\,\Phi\,g^{AB}R_{AB}\left(\Gamma\right)+e^2\int d^{5+1}x\,\Phi\sqrt{-\frac{1}{2\times 4!}g_{AE}\dots g_{DH}W^{ABCD}W^{EFGH}}\;+\\
&-\frac{1}{4!}\int d^{5+1}x\sqrt{-g_{(5+1)}}\,W^{EFGH}\partial_{[\,E}B_{FGH\,]}
\end{aligned} \tag{24}$$

$$\Phi\equiv\epsilon^{A_1\dots A_6}\epsilon_{a_1\dots a_6}\partial_{A_1}\phi^{a_1}\dots\partial_{A_6}\phi^{a_6} \tag{25}$$

where, ϕ^{a_1},… ϕ^{a_6} are six scalar fields treated as independent degrees of freedom and we consider the gravitational action in the first order formulation, i.e. g_{AB} and Γ_{DE}^{C} are treated as independent variables. The connection Γ_{DE}^{C} is torsion-free, i.e. $\Gamma_{DE}^{C}=\Gamma_{ED}^{C}$. Thus, $\partial_{[\,E}B_{FGH\,]}\equiv\nabla_{[\,E}B_{FGH\,]}$ where ∇_M is the covariant derivative. In Eq.(24) $R_{AB}\equiv R^{C}{}_{ABC}$ and $R^{A}{}_{BCD}=\Gamma^{A}{}_{BC\,,D}-\Gamma^{A}{}_{BD\,,C}+\Gamma^{A}{}_{KD}\Gamma^{K}{}_{BC}-\Gamma^{A}{}_{KC}\Gamma^{K}{}_{BD}$.

$\Phi d^{5+1}x$ is a scalar as well as $\sqrt{-g_{(5+1)}}\,d^{5+1}x$ under x-coordinates transformation, while under scalar fields re-definitions:

$$\phi^{a_j}\longrightarrow\phi'^{\,b_k}\left(\phi^{a_j}\right) \tag{26}$$

$$\Phi\longrightarrow\Phi'=J\Phi\;,\quad J\equiv\det\left(\frac{\partial\phi'^{\,a_j}}{\partial\phi^{b_k}}\right) \tag{27}$$

W^{ABCD} = 3-brane slope field, it assigns a tangent (hyper)plane to each spacetime point; the W field is *totally anti-symmetric* in the four indices.
B_{FGH} = 3-brane *gauge potential*; the B field is *totally anti-symmetric* in the three indices. In the last term the invariant integration measure is written in terms of $g_{(5+1)}$, instead of Φ to make the action invariant under (26). One must in this case assume the following Weyl rescalings also

$$g_{A_1A_2} \longrightarrow J\, g_{A_1A_2} \tag{28}$$
$$g^{B_1B_2} \longrightarrow J^{-1}\, g^{B_1B_2} \tag{29}$$
$$g_{(5+1)} \longrightarrow J^6\, g_{(5+1)} \tag{30}$$
$$W^{ABCD} \longrightarrow J^{-3}\, W^{ABCD} \tag{31}$$
$$B_{FGH} \longrightarrow B_{FGH}\ , \qquad \Gamma^{A}{}_{BC} \longrightarrow \Gamma^{A}{}_{BC} \tag{32}$$

Notice that this symmetry holds *only* in the case the embedding space in $6D$. Let us remark that if we define W as a "contravariant" object (upper indices) and B as a covariant field (lower indices), then the last term in the action S depends on the metric only through $\sqrt{-g_{(5+1)}}$.
In a previous paper , [19], a conformally invariant formulation of the brane alone was achieved. This conformally invariant theory concerns branes only. In this paper we are able to formulate the brane plus gravity in a conformally invariant fashion. This is possible if the 3-branes are embedded in a six dimensional space. Of course it is only after the inclusion of gravity into the conformal invariance that the formulation can have an impact into the question of the cosmological constant problem.

We can define the *Dual Representation* of the theory by changing variables

$$W^{ABCD} = \frac{1}{2}\frac{\epsilon^{ABCDEF}}{\sqrt{-g_{(5+1)}}}\,\omega_{EF} \tag{33}$$

$$\begin{aligned} S = \quad & -\frac{1}{16\pi G_{(5+1)}}\int d^{5+1}x\,\Phi\, R_{(5+1)} + e^2\int d^{5+1}x\,\Phi\,\sqrt{\frac{1}{4}\,g^{AE}g^{DH}\,\omega_{AD}\,\omega_{EH}} + \\ & -\frac{1}{6!}\int d^{5+1}x\,\epsilon^{ABCDEF}\,\omega_{[\,AB}\partial_C\, B_{DEF\,]} \end{aligned} \tag{34}$$

In the next section we are going to use the dual formulation, defined by (34), to obtain classical solutions describing a new kind of brane-universe. With this purpose in mind, it is useful to recover the explicit form of ω in terms of the brane density current

$$\begin{aligned} \omega_{MN} \quad &= \frac{1}{4!}\sqrt{-g_{(5+1)}}\,\epsilon_{MNABCD}\,W^{ABCD} \\ &= \frac{1}{4!}\,\epsilon_{MNABCD}\,\delta^{A}_{\mu}\,\ldots\delta^{D}_{\sigma}\,J^{\mu\nu\rho\sigma} \end{aligned} \tag{35}$$

We notice that because of the total anti-symmetry of the ϵ tensor the free indices M, N are actually projected over transverse dimensions. Accordingly, we can write (35) as

$$\begin{aligned} \omega_{MN} \quad &= \rho\,(\,\vec{x}_\perp\,)\;\delta^i_M\delta^i_N\epsilon_{ij}\,\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\int d^{3+1}\sigma\,\delta^{(3+1)}\left[\,x_{||}-Y_{||}(\sigma)\,\right]\,dY^{\mu}_{||}\wedge\ldots\wedge dY^{\sigma}_{||} \\ &= \rho\,(\,\vec{x}_\perp\,)\;\epsilon_{ij}\,\delta^i_M\,\delta^i_N\,\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\int d^{3+1}\sigma\,\delta^{(3+1)}\left[\,x_{||}-Y_{||}(\sigma)\,\right]\,dY^{\mu}_{||}\wedge\ldots\wedge dY^{\sigma}_{||} \\ &= \rho\,(\,\vec{x}_\perp\,)\;\epsilon_{ij}\,\delta^i_M\,\delta^i_N\,\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\int d^{3+1}\sigma\,\delta^{(3+1)}\left[\,x_{||}-Y_{||}\,\right]\,\epsilon^{\mu\nu\rho\sigma}\frac{\partial\left(Y^0_{||}\,,\ldots Y^3_{||}\right)}{\partial\,(\,\sigma^0\,,\ldots\sigma^3\,)} \\ &= -\epsilon_{ij}\,\delta^i_M\,\delta^i_N\,\rho\,(\,\vec{x}_\perp\,) \end{aligned} \tag{36}$$

Eq.(36) shows that ω has non vanishing components only along the transverse dimensions and equals the dual of the brane density.

IV. FIELD EQUATIONS

We will work out the equations of motion in the dual picture first and afterwards we will discuss the brane interpretation of these solutions.
To start let us notice the following facts concerning the action (34). First it can be written in the form

$$S = \int d^{5+1}x\,\Phi\,(\,L_G + L_m\,) - \frac{1}{6!}\int d^{5+1}x\,\epsilon^{ABCDEF}\,\omega_{[AB}\partial_C\,B_{DEF]} \tag{37}$$

where

$$L_G \quad = -\frac{1}{16\pi G_{(5+1)}}\,g^{AB}\,R_{AB}\,(\,\Gamma\,)\ , \tag{38}$$

$$L_m \quad = e^2\sqrt{\frac{1}{4}\,\omega_{AB}\,\omega_{CD}\,g^{AC}\,g^{BD}} \tag{39}$$

are homogeneous of degree one in g^{AC} , that is

$$g^{AB}\,\frac{\partial L_m}{\partial g^{AB}} = L_m\ , \qquad g^{AB}\,\frac{\partial L_G}{\partial g^{AB}} = L_G \tag{40}$$

this property is intimately related to the fact that the action (34) has the symmetry under $g^{AB} \longrightarrow J^{-1}\,g^{AB}$, $\Phi \longrightarrow J\,\Phi$.
The equations of motion which result from the variation of the fields ϕ^a are

$$\mathbf{A}_a^M\,\partial_M\,\,(\,L_G + L_m\,) = 0 \tag{41}$$

where

$$\mathbf{A}_m^M \equiv \epsilon^{MBCDEF}\,\epsilon_{mbcdef}\,\partial_B\,\phi^b\,\partial_C\,\phi^c\,\partial_D\,\phi^d\,\partial_E\,\phi^e\,\partial_F\,\phi^f \tag{42}$$

Since $\det\left(\mathbf{A}_m^M\right) = 6^{-6}\Phi^6/6!$. Then we have that if $\Phi \neq 0$, this means that (41) implies

$$L_G + L_m = M = \text{const.} \tag{43}$$

The equation of motion obtained from the variation of g^{AB} is

$$-\frac{1}{16\pi\,G_{(5+1)}}\,R_{AB} + \frac{\partial\,L_m}{\partial g^{AB}} = 0 \tag{44}$$

by contracting (44) with respect to g^{AB} and using the homogeneity property of L_m, we obtain that the constant of integration M equals zero. Evaluating $\partial L_m/\partial g^{AB}$ and inserting into (44) we find

$$R_{AB} = 4\pi\,e^2\,G_{(5+1)}\,\frac{\omega_{AC}\,\omega_B{}^C}{\sqrt{\frac{1}{4}\,\omega_{MN}\,\omega^{MN}}} \tag{45}$$

Eq.(44) is also consistent with the Einstein form

$$R_{AB} - \frac{1}{2}g_{AB}\,R = -8\pi G_{(5+1)}\,T_{AB} \tag{46}$$

$$T_{AB} = -2\frac{\partial L_m}{\partial g^{AB}} + g_{AB}\,L_m \tag{47}$$

which for L_m is given by

$$T_{AB} = \frac{e^2}{2} \frac{\omega_{AC}\, \omega_B^{\ \ C}}{\sqrt{\frac{1}{4}\, \omega_{MN}\, \omega^{MN}}} - e^2\, g_{AB} \sqrt{\frac{1}{4}\, \omega_{MN}\, \omega^{MN}} \tag{48}$$

as one can easily check that solving from R by contracting both sides of (46) with T_{AB} given by (48) and then replacing R into (46) gives (44).
Let us consider now the equation of motion for the connection coefficients $\Gamma^{A}_{\ \ BC}$. Defining

$$\bar{g}_{AB} = \left(\frac{\Phi}{\sqrt{-g_{(5+1)}}} \right)^{1/2} g_{AB} \tag{49}$$

one can verify that

$$\Phi\, g^{AB} = \sqrt{-\bar{g}_{(5+1)}}\, \bar{g}^{AB} \tag{50}$$

Therefore, the equation of motion for $\Gamma^{A}_{\ \ BC}$ is obtained by the condition that the functional

$$I \equiv - \frac{1}{16\pi\, \pi G_{(5+1)}} \int d^{5+1}x\, \sqrt{-\bar{g}_{(5+1)}}\, \bar{g}^{AB}\, R_{AB}\,(\Gamma) \tag{51}$$

is extremized under variation of $\Gamma^{A}_{\ \ BC}$. This is however the well known Palatini problem in General Relativity (but where the metric $\bar{g}_{AB}$ enters, not the original metric g_{AB}). Therefore $\Gamma^{A}_{\ \ BC}$ is the well known Christoffel symbol, but not of the metric g_{AB} rather than the metric $\bar{g}_{AB}$:

$$\Gamma^{A}_{\ \ BC} = \left\{ {}_{B}{}^{A}{}_{C} \right\} |_{\bar{g}} \tag{52}$$

Notice the interesting fact that $\bar{g}_{AB}$ is conformally invariant, i.e. invariant under the set of transformations (26), (27).
Also, in the gauge $\Phi = \sqrt{-g_{(5+1)}}$, the metric g_{AB} equals the metric $\bar{g}_{AB}$ so one may call this the "Einstein gauge", since here all non-Riemannian contributions to the connection disappear. Alternatively, without need of choosing a gauge one may choose to work with the conformally invariant metric $\bar{g}_{AB}$ in terms of which the connection equals the Christoffel symbol and all non-Riemannian structures disappear. Finally the equations of motion obtained from the variation of the gauge fields ω_{AB} and B_{MNP} are

$$\Phi\, \frac{\omega^{AB}}{\sqrt{\frac{1}{4} \omega_{MN} \omega^{MN}}} = \frac{1}{6!} \epsilon^{ABCDEF}\, \partial_{[\,C}\, B_{DEF\,]} \tag{53}$$

and

$$\epsilon^{ABCDEF}\, \partial_{[\,D}\, \omega_{EF\,]} = 0 \tag{54}$$

taking the divergence of (53) we obtain

$$\partial_A \left(\Phi\, \frac{\omega^{AB}}{\sqrt{\frac{1}{4} \omega_{MN} \omega^{MN}}} \right) = 0 \tag{55}$$

Again, we are looking for solution of (55) where the ω-field is localized over the brane and not propagating into in the bulk. Thus

$$\omega_{EF} = \frac{1}{4!} \epsilon_{EFABCD}\, J^{ABCD} \tag{56}$$

Once (56) is inserted back into Eq.(54) one gets the divergence-free condition for the density current

$$\partial_{_M} J^{^{MBCD}} = 0 \tag{57}$$

which is satisfied because the D–brane is infinitely extended. As one should expect, the classical solution for ω is the dual of the classical solution for W.

V. BRANE-WORLD SOLUTIONS IN THE DUAL PICTURE

In this section we are going to consider the product spacetime

$$ds^2 = g_{\mu\nu}(x_{||})\, dx^{\mu}_{||}\, dx^{\nu}_{||} + \gamma_{ij}\,(\,\vec{x}_{\perp}\,)\; dx^{i}_{\perp}\, dx^{j}_{\perp} \tag{58}$$

where $\mu\ ,\nu\ =\ 0\ ,1\ ,2\ ,3$ and $i\ ,j\ =\ 4\ ,5$. Furthermore, we consider a slope field $W^{^{ABCD}}$ with non-vanishing components only in the first four coordinates $(\,0\ ,1\ ,2\ ,3\,)$, which means we are dealing with a set of parallel branes orthogonal to the extra-dimensions (more on the brane interpretation of the solutions in the next section). This means that the dual field ω_{AB} has non-zero components in the $4\ ,5$ directions only. In this case, we see from eq.(59) that the Ricci curvature induced in the four dimension $0\ ,1\ ,2\ ,3$, is zero:

$$R_{\mu\nu} = 0 \tag{59}$$

Thus, the ordinary four dimensions (accessible to our experience) are not curved by this kind of matter. This is a very important remark, since there is no need to introduce a bare cosmological constant to cancel some contribution from the gauge field, *no type of fine tuning*, most usual in extra dimensional theories, *is needed here*.
The simplest solution of (58) is *flat*, four dimensional spacetime

$$g_{\mu\nu} = \eta_{\mu\nu}\ . \tag{60}$$

Let us analyze now the additional field equations. It is convenient to choose gauge $\Phi = \sqrt{-g_{(5+1)}}$, even if the conformally invariant metric $\bar{g}_{_{AB}}$ gives the same results.
The two-dimensional metric γ_{ij} can always be put in a conformally flat form, i.e. one can always choose a coordinate system where

$$\gamma_{ij}\, dx^{i}_{\perp}\, dx^{j}_{\perp} = \psi\left(\,x^4\ ,x^5\,\right)\left[\,\left(\,dx^4\,\right)^2 + \left(\,dx^5\,\right)^2\,\right] \tag{61}$$

As far as the dual slope field is concerned, its most general form along the extra dimension where it is non-zero, is dictated by its tensorial structure in two-dimensions, which is

$$\omega^{ij} = -\frac{\epsilon^{ij}}{\sqrt{\gamma}}\rho\left(\,x^4\ ,x^5\,\right)\ , \qquad \gamma \equiv \det\left(\,\gamma_{ij}\,\right) \tag{62}$$

It turns out that the field equations do not determine the function ρ as

$$\partial_i\left(\frac{\omega^{ij}\,\sqrt{\gamma}}{\sqrt{-\frac{1}{2}\omega^{kl}\,\omega_{kl}}}\right) = 0 \longrightarrow \partial_i \epsilon^{ij} = 0 \tag{63}$$

which is "trivially" satisfied ϵ^{ij} being the totally anti-symmetric symbol in two-dimensions.
The function $\rho\left(\,x^4\ ,x^5\,\right)$ acts , however, as a source that determines the metric. The physical source of the arbitrariness in ρ can be understood by invoking the brane interpretation of the ω-field. The function ρ is associated to the density of 3-branes being piled in the extra dimensions. Since these branes do not exert any force one upon each other they can be accumulated with an arbitrary density at each extra-dimensional point. Indeed, the definition of ρ by equations

(23) and (62) actually are consistent. Recalling that the scalar curvature of (61) is $R = -\psi^{-1}\nabla^2\psi$, we have from $R = 16\pi G_{(5+1)} L_m$:

$$-\frac{1}{\psi}\nabla^2\psi = 16\pi G_{(5+1)}\,\rho \tag{64}$$

ρ is free to be taken any possible values, but once it is assigned ψ is determined by (64). The argument can be also reversed: for any ψ (64) gives the corresponding ρ. An interesting case is obtained when rho consists of a constant part plus one or more delta function parts. Since R is a scalar a delta function part can appear only in combination $\delta^{(2)}/\sqrt{\gamma}$. Let us define:

$$r = \sqrt{(x^4)^2 + (x^5)^2} \tag{65}$$

$$x^4 = r\sin\phi \tag{66}$$

$$x^5 = r\cos\phi \tag{67}$$

which describe the metric close to $r = 0$, and take $\psi = \psi(r)$, so

$$\gamma_{ij} dx^i_\perp\, dx^j_\perp = \psi(r)\left(dr^2 + r^2 d\phi^2\right) \tag{68}$$

Then, using the representation of the delta-function (with integration measure $rd\phi dr$)

$$\delta^{(2)}\,(\,r\,) = \frac{1}{2\pi}\nabla^2 \ln r \tag{69}$$

where $\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$. Then, for

$$\rho = \sqrt{2}\,B_0 + T\frac{\delta^{(2)}\,(\,r\,)}{\psi} \tag{70}$$

where B_0 and T are constants. By inserting (70) into (64) we obtain (similar equation was obtained in Ref.[22] in the context of $2+1$ gravity)

$$\psi = \frac{4\alpha^2 b^2}{r^2}\left[\left(\frac{r}{r_0}\right)^{\alpha} + \left(\frac{r}{r_0}\right)^{-\alpha}\right]^{-2} \tag{71}$$

where

$$\alpha \equiv 1 - 4G_{(5+1)}T \tag{72}$$

$$b^2 \equiv \frac{\sqrt{2}}{16\pi G_{(5+1)} B_0} \tag{73}$$

Such a metric can be transformed into the form

$$\gamma_{ij}\, dx^i_\perp\, dx^j_\perp = b^2\,\left(d\theta^2 + \alpha^2\,\sin^2\theta\, d\phi^2\right) \tag{74}$$

where ϕ ranges from 0 to 2π, or, equivalently,

$$\gamma_{ij}\, dx^i_\perp\, dx^j_\perp = b^2\,\left(d\theta^2 + \sin^2\theta\, d\bar{\phi}^2\right) \tag{75}$$

where $\bar{\phi}$ now ranges from 0 to $2\alpha\pi < 2\pi$. A complete solution must contain two branes (in the coordinate system (65), (66),(67) we are able to display only one pole of the sphere), the other one is at the other pole of the sphere, where in $(r\ ,\phi)$ coordinates is at $r \to \infty$). Here the term "branes" means delta-functions contributions to ρ.
Of course, this solution is one out of a continuum of solutions, but is interesting because it allows us to connect to other works on the subject (see Ref.[16] where similar effects are discussed).
Nevertheless, we stress the fact that the function ρ is totally free in our model.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have discussed how the gauge formulation of branes can be used in the framework of "brane world" scenarios.

The formulation of 3-branes in a six-dimensional target spacetime can be made in a conformally invariant way. This is possible for extended objects in case the target spacetime has two more dimensions than the extended object itself. This conformal invariance is intimately related to fact that the branes (or equivalently the associated gauge fields) only curve the manifold orthogonal to the brane, the extra-dimensions. No fine tuning of a $6D$ cosmological constant is needed in this case. Therefore, no "old cosmological constant problem" , as Weinberg has defined it [23], appears. An interesting phenomenon is that the parallel 3-branes can be found with an arbitrary density for any value of $\vec{x}_\perp = \left(x^4\ , x^5\right)$. The density $\rho\left(\vec{x}_\perp\right)$ cannot be determined . This represents a large degeneracy and, therefore, a freedom in the possible ways the branes can be accounted in the extra dimensions.

The basic feature, that the matter curves only the extra dimensions is related to the fact that one is able to formulate the theory in terms of the measure Φ, since then Eq.(44) follows automatically. Provided we adopt such formulation Eq.(44) tell us that if L_m depends only from γ_{ij}, then only extra dimensions are curved. Equivalence of this theory to GR requires, however, the existence of conformal invariance since in this case one can choose the gauge $\Phi = \sqrt{-g_{(5+1)}}$, where the gravitational field equations assume the Einstein form. Conformal invariance holds if the embedding space is $6D$.

While a conformally invariant formulation of the brane alone was already achieved in [19], it is only after the inclusion of gravity into the conformal invariance that the formulation can have an impact into the question of the cosmological constant problem. In this paper we are able to formulate the brane plus gravity in a conformally invariant fashion provided the 3-brane is embedded in a six dimensional space.

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